AN APPROXIMATE SOLUTION OF THE PROBLEM OF SEPARATED FLOW PAST AN AXIALLY SYMMETRIC BODY

(PRIBALIZHENNOYE RESHENIYE ZADACHI OF OTRYVNOM OBTEKANII OSESIMMETRICHNOGO TELA)

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If a cavity, generated by separation of flow past a thin axially symmetric body, has an elongated meridian, i.e. if the cavity is "thin", then this flow problem may be solved approximately by the theory of "thin bodies".*

The dimensionless velocity potential for the motion of a body at zero angle of attack may be represented in the form

$$\varphi(x, r, t) = \int_{-\infty}^{+\infty} \frac{g(\xi, t - M\sqrt{(x-\xi)^2 + r^2})}{\sqrt{(x-\xi)^2 + r^2}} d\xi \tag{1}$$

where g is an unknown function which has to be determined from the boundary conditions at the surface of the body and of the cavity, x and r are the nondimensional coordinates in the plane of the meridian, t is nondimensional time, M is the ratio of the characteristic velocity of the body to the velocity of sound in the undisturbed fluid.

Function (1) is the solution of a wave equation, which describes the field of perturbations in the fluid, taking into account the compressibility in linear approximation. It is easily shown [1] that near the axis $r = 0$ there is an asymptotic representation**

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* After the completion of the present work the author became aware of the results of G.V. Logvinovich for the stationary case and zero cavitation number. The equation of a cavity derived by him from other considerations, is identical to the formula derived below for this special case.

** Author's dissertation. Some problems of the hydrodynamics of thin bodies. *MGU*, 1956; references to it are marked by a dagger.

$$\varphi(x, r, t) = -2g(x, t)\ln r + 0(1) \tag{2}$$

Hence it is seen that the main term of this representation does not depend on the compressibility of the fluid. We shall assume the equation of the surface of a body in the form

$$r_1 = \varepsilon f_1(x, t) = \varepsilon f_1[F(t) - x] \tag{3}$$

where we denote by index 1 quantities relating to the surface of the body, $\epsilon \ll 1$ is a small quantity, namely the thickness parameter, $x = F(t)$ is the law of motion of the body nose. On the wetted surface of the body the condition of tangential flow must be satisfied.

$$\varphi_r - \varepsilon (f_1)_x \varphi_x - \varepsilon (f_1)_t = 0 \tag{4}$$

That condition for the function $g(x, t)$ on the segment of the axis of the body, which corresponds to the wetted part of the surface, gives approximately [2]

$$g_1(x, t) = -\frac{1}{2}\varepsilon^2 \dot{F}(t) f_1'[\dot{F}(t) - x] f_1[F(t) - x] \tag{5}$$

In order to find the function $g(x, t)$ on the segment of the axis of symmetry located inside the cavity, the following two relationships must be satisfied in that region:

$$\varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_r^2) = \frac{1}{2}\sigma, \qquad \varphi_r - (f_2)_x \varphi_x - (f_2)_t = 0 \tag{6}$$

Here the unknown function

$$r = f_2(x, t) \tag{7}$$

is the equation of the meridian of the cavity; σ is the cavitation number. In the following the index 2 will denote quantities relating to the surface of the cavity.

Using the asymptotic representation (2) we may replace the complicated integral-differential equations (6) for the functions $g_2(x, t)$ and $f_2(x, t)$ by the approximate relationships

$$\begin{gathered} -2(g_2)_t \ln f_2 + \frac{1}{2}\left\{\left(\frac{2g_2}{f_2}\right)^2 + [2(g_2)_x \ln f_2]^2\right\} = \frac{1}{2}\sigma \\ -\frac{2g_2}{f_2} + 2(g_2)_x (f_2)_x \ln f_2 - (f_2)_t = 0 \end{gathered} \tag{8}$$

For the wetted region of the body surface we have (see equations (2) and (5))

$$f_1 = O(\varepsilon), \qquad g_1 = O(\varepsilon^2) \tag{9}$$

and the functions f_1 and g_1 must transform into functions f_2 and g_2; therefore, the statements (9) are also valid for the functions f_2 and g_2. This indicates that in the relationships (8) the terms containing derivatives with respect to x may be eliminated and then these will become

$$g_t \ln f - \frac{g^2}{f^2} + \frac{1}{4}\sigma = 0, \qquad f_t + 2\frac{g}{f} = 0 \tag{10}$$

where indices have been dropped.

From the first equation (10) it is seen that the approximate relation given here, is valid only for the case where the cavitation number σ is a small quantity, of the order of

$$\sigma = O(\varepsilon^2, \ \varepsilon^2 \ln \varepsilon) \tag{11}$$

The system of ordinary equations (10) is easily integrable in a general form

$$(u')^2 \ln u = 4\sigma (u - u_0), \qquad u = f^2, \qquad g = -\frac{1}{4} u' \tag{12}$$

$$\Phi(u, u_0) \equiv \int \frac{\sqrt{-\ln u}}{\sqrt{u_0 - u}} du = \pm 2\sqrt{\sigma}\,[t + c(x)] \tag{13}$$

where $u_0 = u_0(x)$ and $c(x)$ are constants of integration.

Note that for fixed x the curve (3) has an axis of symmetry ν in the (u, t) plane parallel to the u-axis. The distance between the points of intersection of this curve with the t-axis is

$$T = \frac{a(u_0)}{2\sqrt{\sigma}} \tag{14}$$

therefore for fixed $u_0 \ll 1$ this distance approaches infinity as the cavitation number goes to zero.

The functions $u_0(x)$ and $c(x)$ have to be found from additional conditions related to the transition from the wetted surface of the body to the surface of the cavity. These conditions are derived from the fact that the pressure, the slope of the tangent to the miridian, and the radius of the meridian must be continuous at the cross-section of transition from body to cavity. The location of the cross-section itself has to be determined.

In the case of constant velocity of the body we have $u_0 = \text{const}$, $c(x) = -x + \text{const}$, because in the coordinate system moving with the body, the motion will be steady, namely:

$$u = u(t - x) = u(\xi)$$

From the above it follows, that the cavity in this case will be symmetrical with respect to its mid-section. For $\sigma > 0$ the cavity will have finite dimensions, i.e. will be closed at the finite distance behind the body.

Further, from (14) it follows that for $\sigma \to 0$ and $u_0 = \text{const}$, the length of the cavity increases indefinitely. All this is in agreement with the conclusions obtained from the exact solution of the problem for these various conditions as far as they are available. Exact solutions however are known only for the plane problem.

Investigations of the asymptotic representation $u(\xi)$ for $u \to 0$ shows that the cavity at the downstream point of its termination on the axis is a smooth surface, i.e. $f'(\xi) \to \infty$ as $f \to 0$. Obviously, the given approximate solution ceases to be valid in the neighborhood of this point.

The conditions valid at the transition cross-section located at the unknown coordinate ξ_1 are as follows:

$$\left[(\ln u_1)\, u_1'' + \frac{1}{2u_1}(u_1')^2\right]_{\xi=\xi_1} = \left[(\ln u_2)\, u_2'' + \frac{1}{2u_2}(u_2')^2\right]_{\xi=\xi_1} = 2\sigma \qquad (15)$$
$$u_1(\xi_1) = u_k(\xi_1 + c,\, u_0), \quad u_1'(\xi_1) = u_2'(\xi_1 + c,\, u_0)$$

These three relationships serve to determine the three undetermined parameters ξ_1, u_0, c. In the case when the cavity is created by a separation at a break of the body meridian the pressure at the point of separation, in general, will have a discontinuity.

The general solution found here allows in principle the solving of the problem of flow with separation for small positive cavitation numbers for an arbitrary unsteady motion of the body. Besides it may be applied to the solution of penetration of a thin body into a semi-space folled with a fluid at rest when behind the penetrating body a free surface is created. (For the case when separation does not occur the problem is solved in a previous work*). In this case the law of motion of the body may also be arbitrary, though the angle of attack must be zero; the angle between the axis of the body and the free surface may also be arbitrary. All this follows from the properties of the solution of the problem by the theory of "thin bodies"n*

In the case of penetration, obviously, σ must be taken as zero. The solutions of equations (12) for $\sigma = 0$ approach $u = 1$ for finite t, also on the line $u = 1$ of the (u, t) plane the integral curves have a cuspidal point. These facts render the solutions obtained invalid for $\sigma = 0$ and t increasing, because the original assumptions of the theory are no longer true. It was assumed that u be small ($u \ll 1$); however, in this instance u is of the order of unity.

Near the body the behavior of the cavity is the same for the cases $\sigma > 0$ and $\sigma = 0$, because asymptotic relations $u(t)$ for $u \to 0$ are the same (see (12)).

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